

# Enhancing Mathematics Understanding through Visualization:

## The Role of Dynamical Software

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# Chapter 6

## Vectors and Differential Equations:

### A Visual Approach using Autograph

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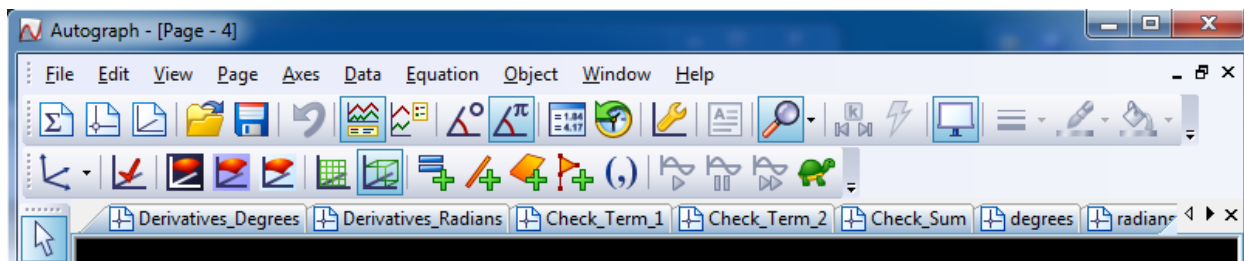
#### ABSTRACT

*This chapter is inspired by a session the author gave at the 10th EMAC Conference (Engineering, Mathematics, and Applications) at the University of Technology, Sydney, in December 2011. The audience was university teachers, but the software, Autograph, was designed for use in High Schools. The author was able to show how a simple, pedagogically focused interface could be used to create a highly visual approach to the teaching of two favourite topics: Vectors (in 2D and 3D) and Differential Equations (1<sup>st</sup> and 2<sup>nd</sup> Order).*

#### INTRODUCTION

Teachers of University mathematics are generally used to using one of the software giants, such as Maple, Mathcad, or Mathematica. These tools can be tamed to perform almost any mathematical task, but each requires a detailed knowledge of syntax; furthermore, the presentation is generally in "notebook" style, which is perfect for research use, but can appear confusing when used as visualizer in the

*Figure 1 Autograph has a cleanly designed menu/icon system for its statistics, 2D and 3D pages*



classroom. Autograph on the other hand is designed with teaching in mind: the screen is uncluttered, and all operations are easily found by the judicious use of ‘select’ and ‘right-click’.

The University teachers in Sydney were somewhat surprised that software created for high school use could bring so many new insights into the teaching of vectors and differential equations.

## BACKGROUND

Autograph was conceived in the secondary classroom, covering topics right up to University entrance in the UK. This included many topics that have long since risen above school curriculums in the general rush to consolidate, together with the laudable aim of trying to encourage more students to study mathematics. The two topics of this chapter, vectors and differential equations, used to be well inside the school envelope, but they now occupy the overlapping space between the top end of high school and the first year of university mathematics.

In the design of Autograph, there was an overriding determination to make the creation of dynamic objects as straightforward as possible. This has led to an environment where it is possible to study concepts from the “bottom up”. The logical addition of dependent objects and variable parameters can help students to build firm foundations based on a creative mix of dynamic visualisations and solid bookwork.

To get the most out of this article, you are invited to download and install Autograph (PC or Mac) from

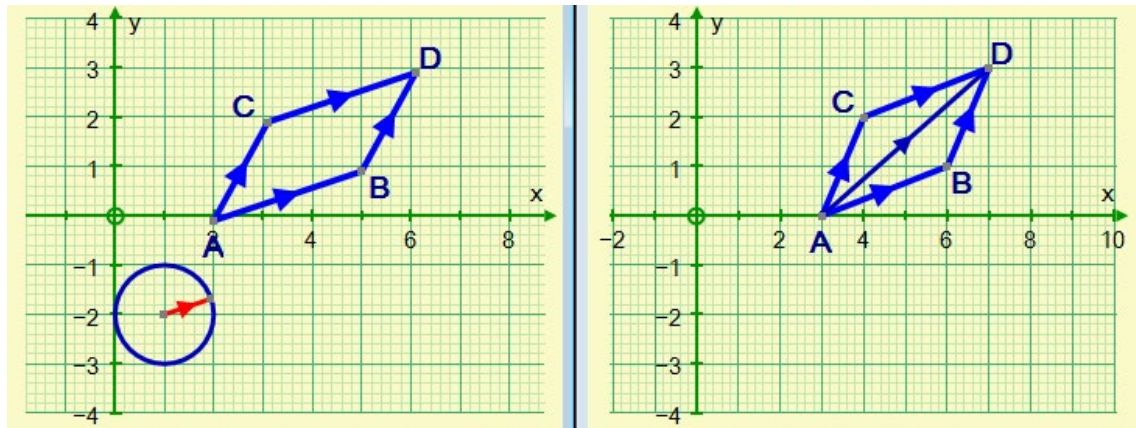
[www.autograph-maths.com/download](http://www.autograph-maths.com/download)

and to email the author for a serial number (otherwise your installation will run out after 30 days).

Please also refer to the online dynamic full colour version of this chapter on the Web:

[www.tsm-resources.com/dynamic](http://www.tsm-resources.com/dynamic)

Figure 2. A clear understanding of elementary operations can be gained if the vectors move



## Representing vectors in 2D

I am a firm believer in teaching vectors in 2D the way they are handled in 3D. This way the transition from 2 to 3 dimensions can be painless.

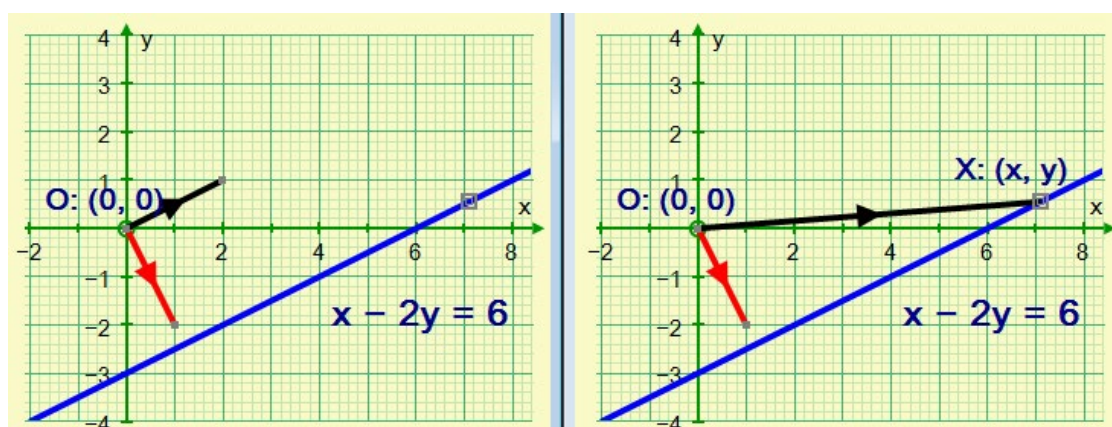
Figure 2 illustrates a strategy for understanding the principles of vectors from a standing start. In Autograph you can define a vector on a given point by entering its two components, e.g. define  $AB = [3, 1]$ , and the point "A" is moveable, or define the vector between two points, e.g. A (3,0) and B (4,2), in which case both A or B are moveable.

Moving the point A shows that the vector  $[3, 1]$  can in fact be anywhere, including copied onto C (to form CD). Likewise, AC can be copied to BD. Hence, the parallelogram law can be studied, but only once we have established addition. The addition of two vectors is defined as a vector with the sum of the x and the sum of the y components, and hence can be represented by the leading diagonal of the parallelogram.

To move on to subtraction, we recall that in simple addition of numbers,  $3 + (-3) = 0$ : so the process of subtraction requires the concepts of a negative number and of zero to be in place. So it is with vectors: to establish subtraction, we need a "negative" vector and "zero" vector to be defined. Students can be asked to move C until the sum is zero. This leads to the definition of a negative vector and a zero vector, and ultimately to the realization that subtraction can be represented by the other diagonal of the parallelogram.

The concept of a unit vector can also be explored: draw a unit circle on a point somewhere, and create a unit vector on that point from the vector AB (or CD). Note that it is in the same direction, but of unit length.

Figure 3. A dynamic representation of the vector equation of a straight line



There are many forms for the straight line: explicit (e.g.  $y = 3x - 1$ ) and implicit (e.g.  $x - 2y = 6$ ), and based on the vector form:  $[x, y] = [a, b] + k[l, m]$ , which is a line through  $(a, b)$  parallel to the vector  $[l, m]$ .

It is useful to use the implicit form to prepare students for the equation of a plane, and to establish that the vector  $[a, b]$  is perpendicular to the line  $ax + by = c$  (as illustrated in Figure 3 on the left). With this straight line, the parallel vector is  $[2, 1]$ , and so the perpendicular vector is  $[1, -2]$ . Is it a coincidence that "1" and "-2" are the coefficients of "x" and "y" in the straight line equation?

To confirm this, place a point "X" on the line. The vector from the origin to X is the vector  $[x, y]$ . It is useful to be able to confirm that the scalar product of  $[1, -2]$  and  $[x, y]$  is 6. More generally, for the line  $ax + by = c$ , the value of  $[a, b] \cdot [x, y]$  is c and  $[a, b]$  is perpendicular to the line (as illustrated in Figure 3 on the right)

### The Transition to 3D

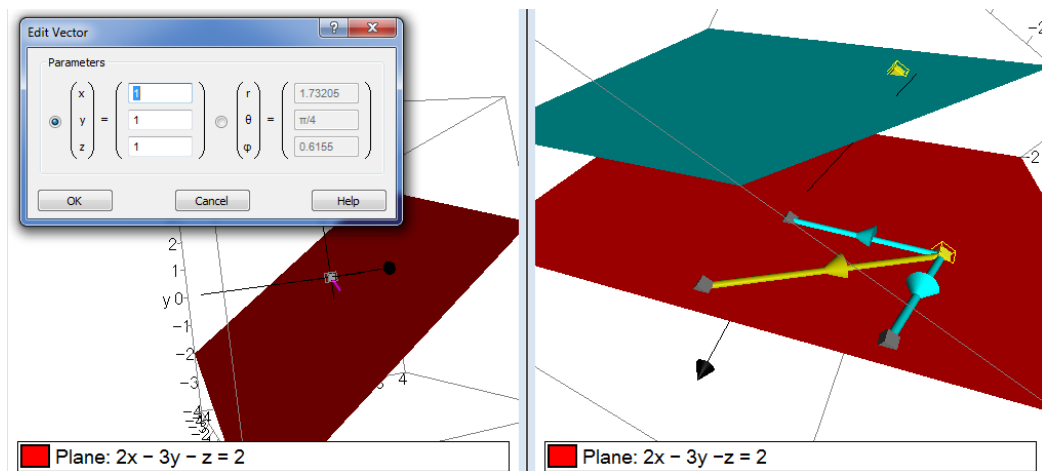
The transition to 3D should now be a simple matter of adding another dimension. A line becomes a plane and the perpendicular vector now takes its components from the three coefficients of "x", "y" and "z" (Figure 4, left).

Figure 4, right, shows how three points in a plane can create two vectors whose sum also lies in the plane. Furthermore, moving the original vectors around the plane can demonstrate how any point in the plane can be found as a linear combination of the original two vectors. Hence, a plane can be defined from a point and two vectors, for example the upper plane in the diagram.

### The Shortest Distance between Two Skew Lines

This is a tricky topic to teach, but one that responds beautifully to a dynamic visual approach. In Figure 5, left, a straight line has been formed from two

Figure 4. The extension from 2D to 3D is a natural step with dynamic figures



points, and a second line entered in vector form, through  $(1, 2, 0)$  and parallel to the vector  $[-1, 1, 1]$ . The shortest distance is part of a line that is in a direction that is mutually perpendicular to the two lines. To illustrate this, create two vectors, one on each line. Place a point somewhere on one of the lines, and create on this point the cross product of the two vectors.

Now construct a straight line through the end points of this vector. In Figure 5, right, this point has been moved along the line on which it is based until the perpendicular line cuts the second of the two original lines. This new point defines the shortest distance! You can confirm this by getting Autograph to do the calculation: select the two original lines and ask for "shortest distance".

### The Line of Intersection of Two Planes

In Figure 6, left, one plane is  $2x - 3y = 6$ , and the second is defined by a point  $(1, 1, -1)$  and the two vectors  $[1, -1, 1]$  and  $[2, 1, 1]$ . Autograph has been asked to construct the line of intersection, and in doing so states its vector equation as

Figure 5

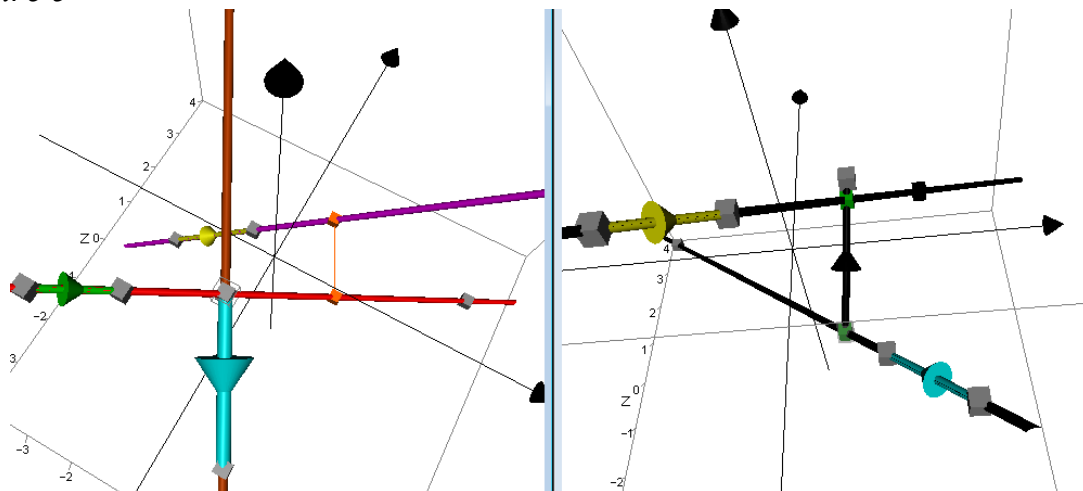
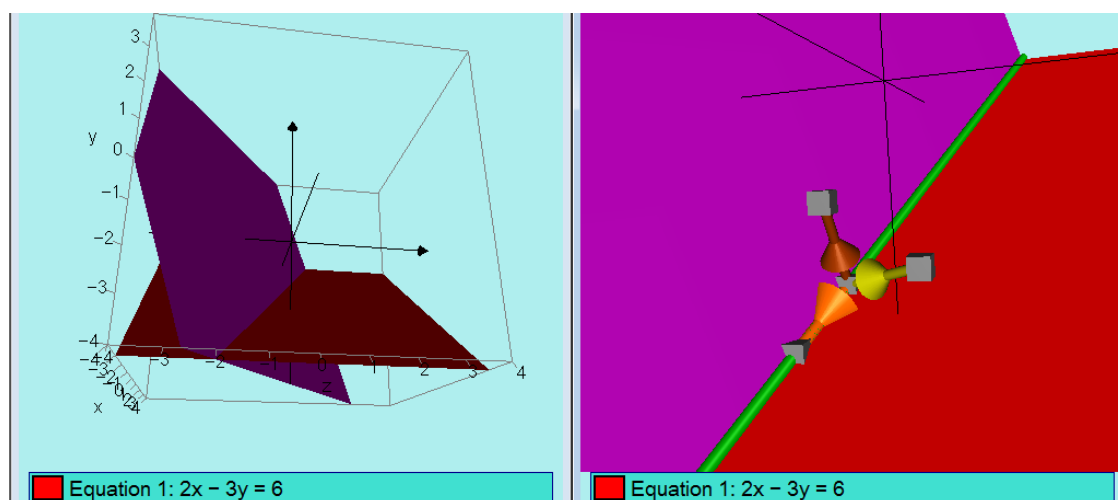


Figure 6



$[x, y, z] = [0, -2, -0.6667] + \lambda[0.7804, 0.5203, 0.3468]$   
and the angle between the two planes is  $58.74^\circ$ , or 1.025 radians.

This version of the vector equation is of course only one of an infinite number of possibilities, based on any point on the line, and any vector that is parallel to  $[0.7804, 0.5203, 0.3468]$ , so this is unfortunately not a good vehicle for checking student answers!

So far so good, but where is the understanding?

If you place a point someone along the line, and then for each plane construct the normal unit vector, you have the opportunity to draw the cross product of these

two unit vectors. Autograph gives the answer for this cross product as  $[-0.6671, -0.4447, -0.2965]$ . Is this vector parallel to the vector given as part of the intersection line equation:  $[0.7804, 0.5203, 0.3468]$ ?

This will demonstrate convincingly that the line of intersection is in a direction that is mutually perpendicular to the two planes, as shown in Figure 6, right.

### While We Are in 3D ...

Other topics to benefit from 3D graphing include sections of a cone, and various possibilities in the construction of volumes of revolution of an area about an axis.

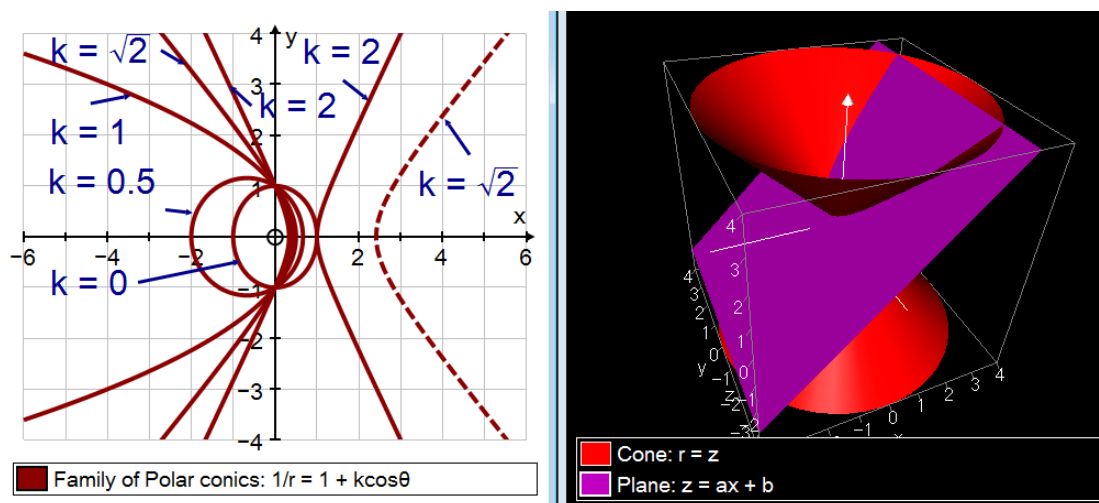
The teaching of all these topics has been immeasurably improved by software that can treat them in 2D and 3D using dynamic objects.

### Visualising Conic Sections

In Figure 7, left, a family of conics has been drawn from the polar equation  $1/r = 1 + k \cos \theta$



Figure 7



with values of "k" set as follows:  $k = 0$  (circle),  $k = 0.5$  (ellipse),  $k = 1$  (parabola),  $k = \sqrt{2}$  (rectangular hyperbola) and  $k = 2$  (hyperbola).

A good knowledge of these important curves in 2D can be achieved through the polar form, though a firm understanding of all the Cartesian forms and their various coordinate geometry properties is also important.

With this grounding in 2D, the 3D scene is a perfect follow-on. Figure 7, right, shows the polar form of a cone,  $r = z$  (though the Cartesian form  $x^2 + y^2 = z^2$  is another way of plotting the same thing). So how to draw the intersecting plane? In 2D,  $y = mx + c$  is a straight line. In 3D the same equation is a plane as it plots  $y = -mx + c$  "for all  $z$ ". The plane that we want is  $z = ax + b$  (plotted "for all  $y$ "), and varying "a" is just the same as varying "m" as the slope of the line in 2D. Autograph's constant controller can vary "m" to find all the categories of conic section, including of course the double-branched hyperbola.

### A New Look at Volumes of Revolution

In Figure 8, a parabola has been drawn (plotted as 2D in 3D), and an element of area created between two points on the curve that are close to each other, which is just a single rectangle. In Figure 8, left, an axis of rotation has been drawn from  $y = -1$  (treated as a 2D object in 3D), and in Figure 8, right, the axis is  $x = -1$ .

In each case an element of volume is shown, but how important it is to train students to predict these shapes before they are drawn.

Some wonderfully interesting 3D volumes can be created using areas, appropriate axes of revolution, and different "rules" for calculating the area. How important a good grounding in 2D is before tackling the 3D situation. In Figure 9, left, the process of finding the area between two curves is being explored in 2D, and in Figure 9, right, the effect of rotating such an area to form a solid in 3D.



Figure 8

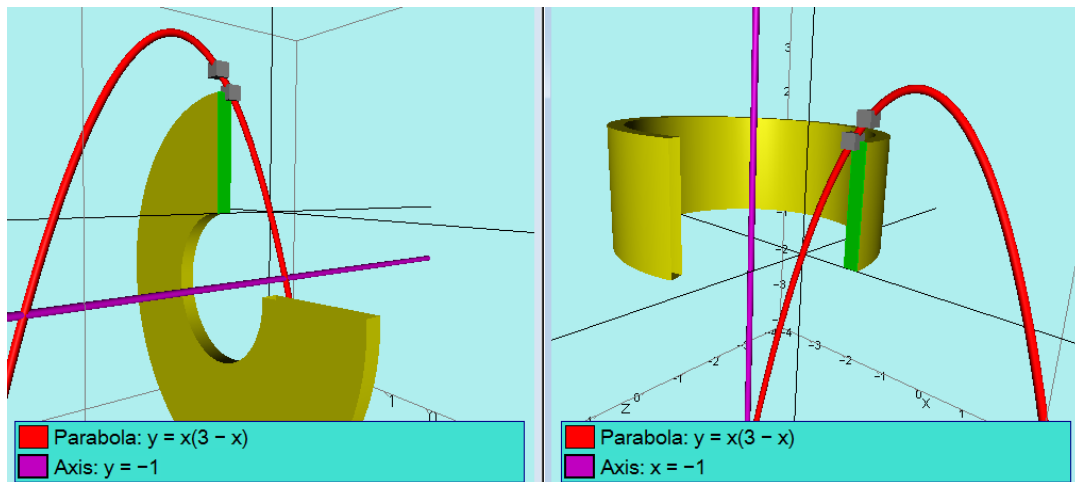
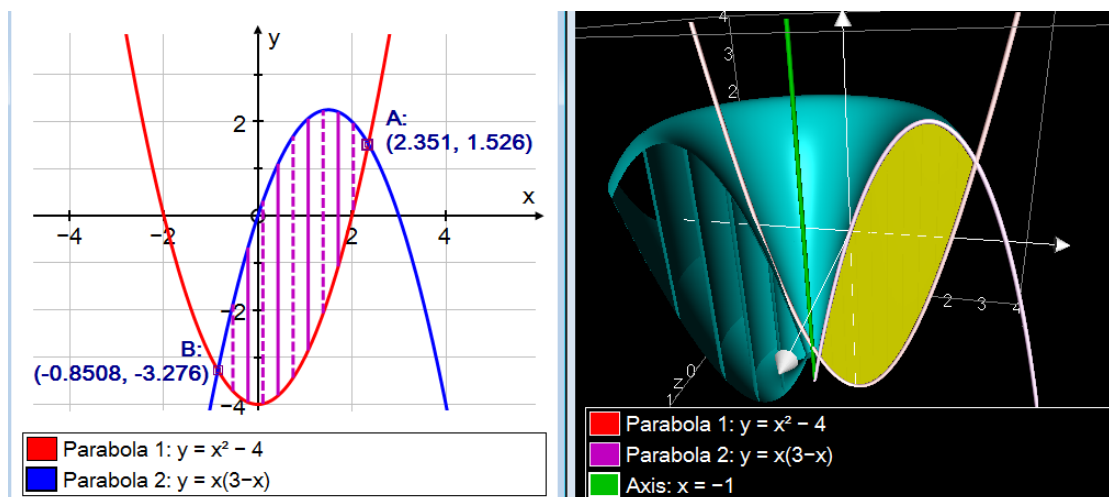


Figure 9 Similarly, the extension of 2D to 2D in calculus is a natural step



### Teaching 1<sup>st</sup> Order Differential Equations from the bottom up!

This has to be a topic that can really benefit from the use of dynamic images. Additionally, there are big advantages for teachers and learners of differential equations to having software available that will instantly plot slope fields from equations that are entered either explicitly or implicitly.

With Autograph, it was decided to make use of the keyboard characters ' and " to create  $y' = dy/dx$  and  $y'' = d^2y/dx^2$  entries. In the case of "t" being the independent variable and "x" the dependent variable, characters x-dot and x-double-dot [ $\dot{x}$  and  $\ddot{x}$ ] have been used.

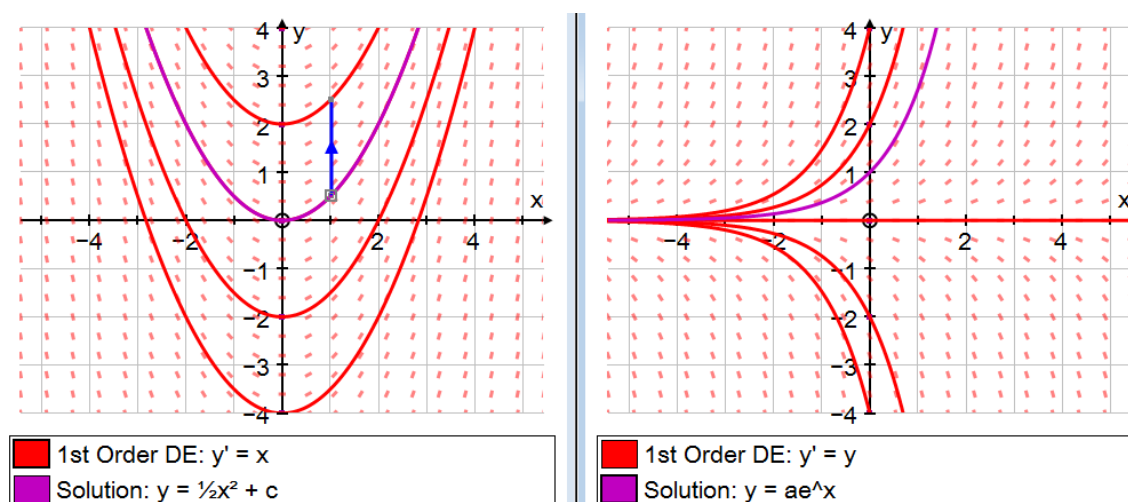
The other interface issue to settle is how to enter the starting conditions. This requires one or more points to act as the starting point for the solution. With Autograph there are three options:

- Start each solution off by a mouse click.
- Use a selected point as the start point, which can then be dragged around.
- Establish and use a set of points on a line which will create a neat family of solutions.

With a tool such as Autograph it is possible to establish a really firm understanding of the basic principles by tackling the subject from the "bottom up". A typical sequence to explore the basic linear first order differential equation is:

1.  $y' = 0$   
The solution set is many horizontal straight lines.
2.  $y' = 1$   
The solution set is many lines straight lines with gradient = 1.
3.  $y' = x$   
This yields is a family of parabolas. You can then compare the slope field with a family of parabolas from  $y = \frac{1}{2}x^2 + c$ , as in Figure 10, left. In this diagram, a moveable vector  $[0, 2]$  has been placed at a point on one of the curves, to show that it is a vertical displacement that defines the family of parabolas.
4.  $y' = y$   
Ask what the solution is, and hold your breath as answers such as “parabolas the other way” come in - it may have been only last week you were teaching the exponential function! See Figure 10, right.  
This is of course a nice reminder that  $y = e^x$  is a function that is equal to its own derivative, but what are all the other solutions indicated by the slope field? Again, wait for “ $e^x + c$ ” ... ! The visual approach confirms that the answer cannot be  $y = e^x + c$ . So where does the constant of integration go?
5.  $y' = -y$   
This is the first time an exponential decay appears. So now we can usefully rearrange implicitly:
6.  $y' + y = 0$   
This is also the exponential decay of course, and the basic building block of the whole topic of linear 1<sup>st</sup> and 2<sup>nd</sup> order differential equations.
7.  $y' + y = x$   
The left side of this equation is referred to as the “complimentary function”; the right side is the “particular integral”. This visual approach confirms that this complimentary function always implies an exponential decay down to the steady state, in this case  $y = x - 1$ . See Figure 11, left.

Figure 10. Introducing 1<sup>st</sup> order differential equations “from the bottom up”



### 8. $y' + y = \sin x$

This is another good example of exponential decay, this time decaying to a periodic function, see Figure 11, right, which also shows the relationship between the steady state and the particular integral.

Helping students to predict slope fields is as important as it is to ensure that they predict regular graphs before they are computer-drawn. Figure 12 gives two good examples of predictable outcomes.

## Other Applications of 1st Order Differential Equations

There are many other applications of 1<sup>st</sup> order differential equations that can be visualized well; here are two more. First, the plotting of isoclines (points of equal slope), in this case, in Figure 13, left, the family of circles that are related to the differential equation  $y' = x^2 + y^2$ .

The second example, in Figure 13, right, is the modeling of a falling object that is approaching its terminal velocity. In this case the independent variable is time ( $t$ ) and the dependent variable is velocity ( $v$ ). The equation is derived from applying Newton's 2<sup>nd</sup> Law to the falling mass:

$$mg - kv^2 = m \, dv/dt$$

and the appropriate constant values entered ( $m = 80\text{kg}$ ,  $g = 9.81 \text{ m/s}^2$ ,  $k = 2$  and  $n = 2$ ). Again, the ability to enter a differential equation implicitly is a real benefit.

Figure 11. How to visualise the complimentary function and particular integral

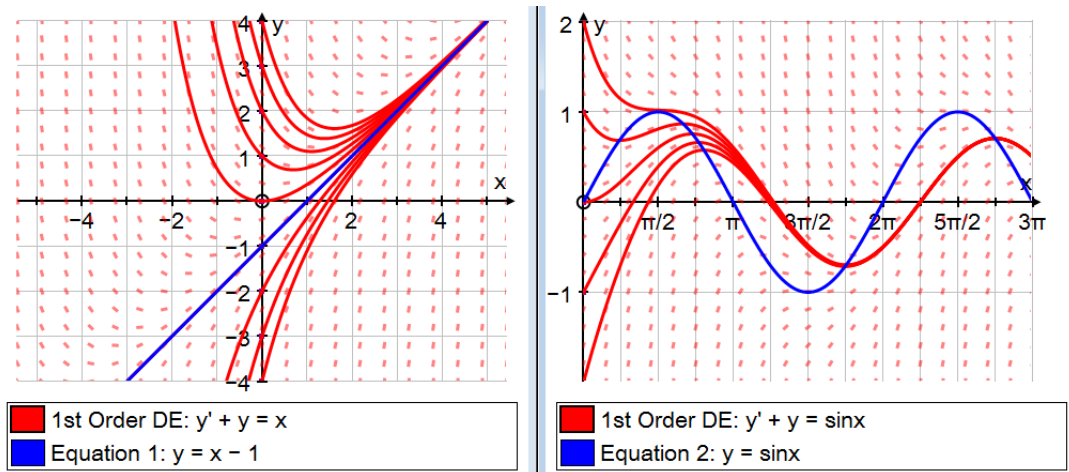
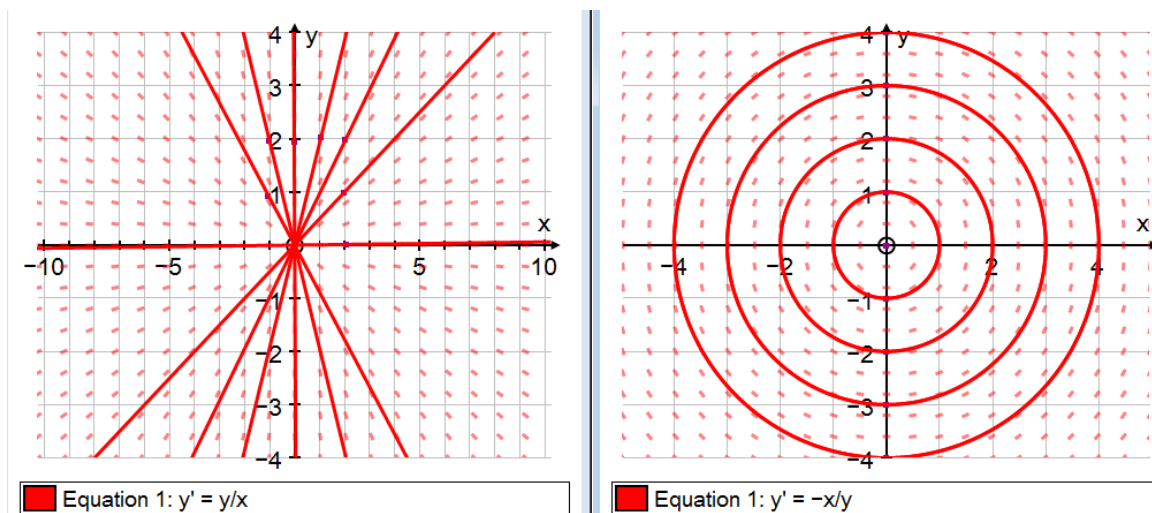


Figure 12. How to predict special instances of elementary 1st order D.E.s



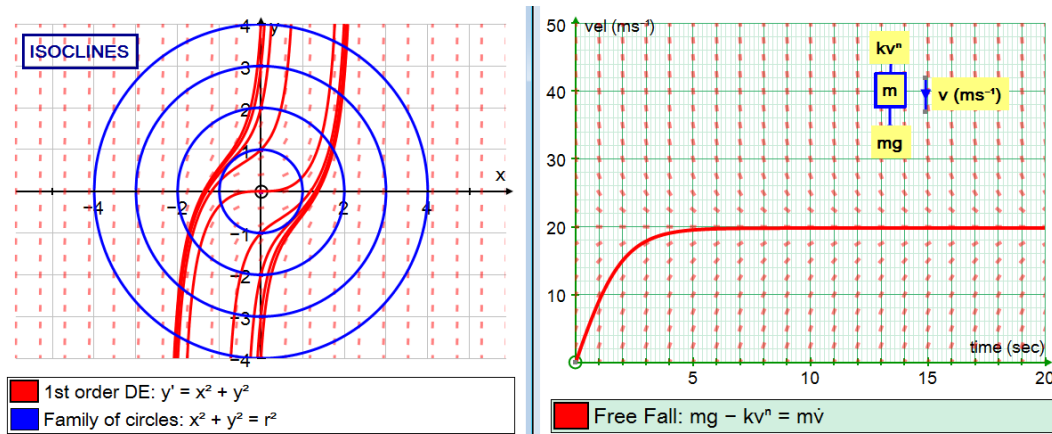
### Teaching 2<sup>nd</sup> Order Differential Equations from the Bottom Up!

If you can establish a firm understanding of graphing and vectors in 2D then the 3<sup>rd</sup> dimension will follow naturally. Similarly, if you work through 1st order D.E.s systematically, and using the implicit form, then the move to second order equations can be painless. Once again, the ability to enter differential equations implicitly is of paramount importance to a firm understanding of their form.

Second Order equations require two starting conditions. Autograph offers the chance to define the first as one or more points at the start, with the same options as the 1<sup>st</sup> orders (click, selected point or a line of points). The second condition is determined by the initial slope.

This is a typical sequence for “bottom” up learning of the 2<sup>nd</sup> order equation:

Figure 13



1. Reset the axes to the “x” on the vertical axis and “t” on the horizontal axis, taking account of the fact that the solutions of 2<sup>nd</sup> orders are usually concerned with displacement and time
2.  $\ddot{x} = 0$   
On the first integration, we get  $\dot{x} = a$ , and on the second integration we get  $x = at + b$ . ie The solutions take the form of any straight line through the start point with any initial gradient.
3.  $\ddot{x} = 1$   
On the first integration, we get  $\dot{x} = t + c$ , and on the second we get  $x = \frac{1}{2}t^2 + ct + d$ .  
ie Any parabola through the start point with any initial gradient.
4.  $\ddot{x} = x$  [see Figure 14, left]  
The interpretation here is that the acceleration is away from the t-axis whether “x” is positive or negative. The solution takes the form  $x = a \exp(t) + b \exp(-t)$ .
5.  $\ddot{x} = -x$   
Here the acceleration is always towards the t-axis, and switches as a solution crosses the t-axis, hence there are oscillations which turn out to be sinusoidal. Figure 14, right shows many of these DE forms, all on the same pages, all starting at (0,1) with zero initial gradient.
6.  $\ddot{x} + x = 0$   
This is a literal rearrangement of  $\ddot{x} = -x$ , so a sinusoidal solution is expected. In Figure 15, left, the plot of  $\dot{x}$  has also been included (as a dotted line).
7.  $\ddot{x} + x = f(t)$   
This gives the opportunity to explore many different particular integrals
8.  $\ddot{x} + 2\lambda \dot{x} + n^2x = 0$   
This introduces “damping”, and the constant controlled is a good way to vary “λ” and hence vary the damping (Figure 15, right).
9.  $\ddot{x} + 2\lambda \dot{x} + n^2x = f(t)$   
This gives the opportunity to explore many different particular integrals.

Figure 14 Elementary forms of 2<sup>nd</sup> order differential equations

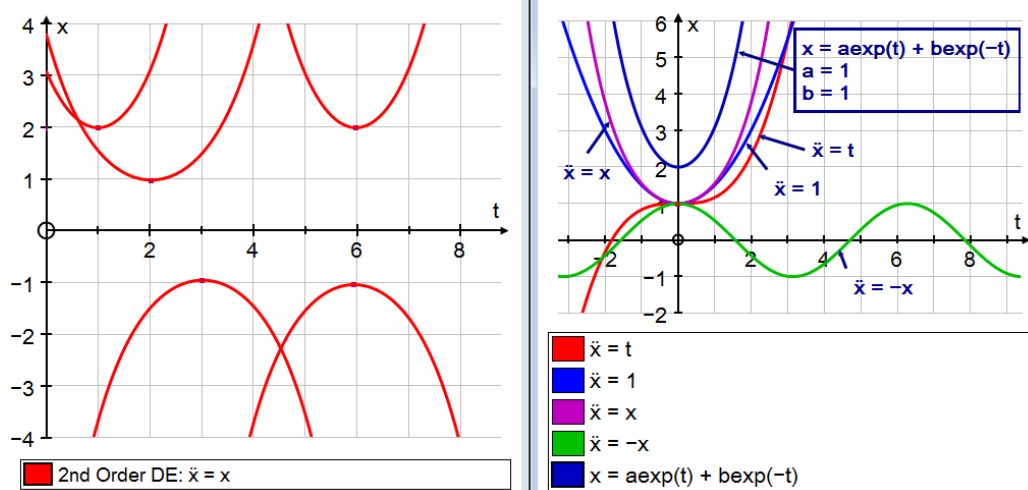
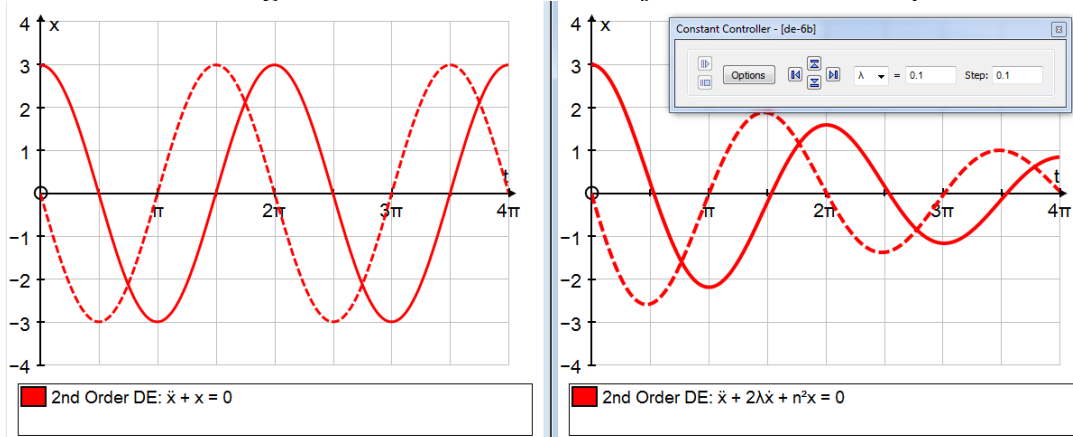


Figure 15. Introducing 2<sup>nd</sup> order linear D.E.s “from the bottom up”



## Entering Mathematical Notation, when One-Line Mathematics is Sufficient

There are many occasions when mathematical expressions can be perfectly adequately entered into “Word” as a single line using the standard “Unicode” symbol. The equations for this article were all entered using the Autograph onscreen keyboard:



## CONCLUSION

There are many software systems available to take these and many other topics on to a high level, but the approach taken by Autograph is ideally suited to establishing good understanding from the bottom up. Mathematics is, above all, a subject that requires a firm understanding of basic principles. All teachers and students should be wary of any attempt to suggest that modern software and Internet resources can in any way short circuit the learning process.

Autograph was designed for the visualisation of school-level and first year university level topics. The two topics in this chapter, vectors and differential equations, are perfect examples of how successful a visual approach can be.

However, a printed chapter is not the ideal environment in which to discuss the impact of dynamic software – its power emanates almost entirely from the fact that things move! There is a dynamic Web page dedicated to illustrating all the examples in the chapter on [www.tsm-resources.com/pubs/dynamic-ch6](http://www.tsm-resources.com/pubs/dynamic-ch6)

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Oundle, March 2012

## REFERENCES

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|---------------------------------|--|
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| Douglas Butler's Blog           | <a href="http://dougbutler.wordpress.com">http://dougbutler.wordpress.com</a>                      |
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